The Young's modulus made easy

1 Preamble.

The Young's modulus is a constant which describes the flexibility of a material. It is well known by string instrument makers as the main factor contributing to string inharmonicity. Another less known factor is the longitudinal string vibration frequency which plays a role in the sound colour and also directly depends on the Young's modulus. Fortunately, the longitudinal modes also provide us with a yet unpublished, cheap, non-invasive method to measure the Young's modulus of an existing string in an instrument. The aim of this paper is to describe how to do this measurement, and how to use the result to recognise the material of the string and calculate the string's inharmonicity.

2 Introduction.

Let us begin with an experiment. With a little rosin at a fingertip (or on a glove if required), gently rub a string lengthwise – on a pianoforte, clavichord or harpsichord it is easiest on medium or bass strings. Of course, the corresponding damper must be raised during the operation. The string will vibrate, usually producing a high pitched, fluted sound; this is the fundamental longitudinal mode.

Unlike with the usual (transverse) vibration mode, the frequency of the longitudinal mode does not depend on string tension. If in doubt, have a try and slightly detune the string: the longitudinal sound will remain unchanged.

The classical formula[1] giving the fundamental transverse vibration frequency of a string is

$$F_t = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

with

- F_t frequency,
- L speaking length,
- T string tension,
- μ mass per unit length.

Similarly, the fundamental longitudinal vibration frequency F_l is given by:

$$F_l = \frac{1}{2L}\sqrt{\frac{E}{\rho}}$$

with

- ρ material density,
- E Young's modulus.

which confirms the longitudinal frequency does not depend on the string tension.

3 Measuring the Young's modulus without expensive tools.

The formula above may be written:

$$E = 4\rho L^2 F_l^2$$

and can be directly used to measure *in-situ* the Young's modulus of an existing string.

I tried this on the centre C (c'/C4) of my 1914 Erard grand, which still retains its original strings. Its speaking length is 680mm. The rosin test gives a note three octaves plus one sixth above the normal (transverse) c' of this string, i.e. a''''/A7. An easy calculation shows this

[1] See for example: Fletcher (Neville H.), Thomas D. Rossing, *The Physics of Musical Instruments*, 2nd edition, Springer1998-99.

corresponds to a frequency of 3520 *Hz*. The strings visually appear to be made from iron or steel, with a density around $7770\,kg/m^3$. Following the preceding formula, the Young's modulus of this string is therefore

$$E = 4 \times 7770 \times 0{,}68^2 \times 3520^2 \approx 1{,}78 \times 10^{11} \approx 180\,GPa$$

which is perfectly typical of iron rather than steel. More generally, common values of the Young's modulus are around:

- 300 *GPa* for steel
- 200 *GPa* for iron
- 120 *GPa* for brass
- 80 *GPa* for red copper

As we can see here, relatively recent pianofortes, even overstrung and fitted with a cast frame like this one, often carry iron strings.

In practice this method gives the Young's modulus with an accuracy around 10%, which is sufficient in most cases. The main difficulty is not to get the wrong octave, which would anyway lead to aberrant values. Luckily enough, the Young's modulus is not very much affected by wire drawing[2].

4 Inharmonicity.

Inharmonicity refers to the shift of the partials relative to their theoretical values as multiples of the fundamental frequency. Inharmonicity is driven by the *basic inharmonicity factor*, which is for a plain (unwound) string:

$$B = \pi^3 \frac{E\,d^4}{64\,T\,L^2}$$

where

- E is the Young's modulus (only depending on the material);
- d is the string diameter;
- T is the tension;
- L is the speaking length.

The actual value of each partial is:

$$f_n = n\,F\left(1 + B\,n^2\right)^{1/2}$$

On an ideal, perfectly flexible string, $B = 0$ and the partials are the exact multiples of the fundamental frequency. On real strings, the material's stiffness tends to stretch the intervals between partials, in particular the octaves: rather than having $f_2/f_1 = 2$, we have

$$f_2/f_1 = 2\,F\,(1 + 4B)^{1/2} / F\,(1 + B)^{1/2} \approx 2\left(1 + 1.5\,B\right)$$

which means the second partial is slightly more than one octave above the fundamental. On a harpsichord or clavichord, inharmonicity is only significant in the bass and is very low in the medium and treble anyway. On most pianofortes inharmonicity is higher but similar and only significant in the bass; while most 20th Century pianos also have noticeable inharmonicity in the two treble octaves, inharmonicity is still very low in the middle register and a change in the Young's modulus would not modify it significantly. In all these cases, inharmonicity is not able to account for the proven differences in sound quality between different string makes throughout the instrument's compass. Longitudinal vibration is one of the factors which probably play a role here.

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2 Goodway (Martha) and Jay Scott Odell (eds.), *The metallurgy of 17th and 18th century music wire*, in *The Historical Harpsichord*, volume two, Pendragon Press 1987.

5 Sound colour, longitudinal modes and the Young's modulus.

On a stringed keyboard instrument, the longitudinal modes cannot be neglected:

- On a piano, the hammer flanges are never in the strings' plane. This means that even if a hammer head is perpendicular to the string it is hitting, it will not hit the string perpendicularly but obliquely. which sets the longitudinal mode into vibration.
- On a harpsichord, as the jack motion is normally vertical, there is no initial excitation to the longitudinal modes. However, as on any stringed instrument, there is a coupling between the transverse and longitudinal modes: the response of the bridge to a vertical excitation is not vertical but oblique, depending on the soundboard and bridge geometry; moreover, the string's tension oscillates with twice the frequency of the transversal vibration, which again excites longitudinal vibration modes.

All of the above, along with the neighbouring strings and 'aliquot segments', contributes to the colouring of sound through what is called the *formants*.

When we are speaking, our vocal cords produce a vibration similar to singing, the fundamental frequency of which is called the *pitch*. The shape of the mouth and the tongue's position allow to modulate the relative power of the different parts of the spectrum and favour some parts of the spectrum, called the *formants*. The positions of formants define which vowel is being pronounced, independently of the pitch. Thus, for example the vowels E and O are characterised by different positions of the formants; if the same vowel is sung and the pitch is changed, the formants will remain the same and the peaks of the spectrum will move inside the formants which will be acting as envelopes. A man and a woman producing the same vowel but probably at different pitches, will use the same formants with the same absolute positions. On a spectrogram, a vowel is not characterised by its pitch or its balance of harmonics, but by its formants, i.e. the envelope which encloses these harmonics. Singing a gamut on a given vowel will result in having the spectrum moving but remaining inside this envelope.

On the other hand, when using an old fashioned analogue tape recorder, changing the tape speed moves all the frequencies the same way and the absolute position of formants will then be altered: the sound O may become E and the speech will soon become unintelligible, not because it is spoken fast but because the formants are changed.

In a stringed instrument, the longitudinal modes as resonators participate the same way to the production of formants which give the sound its colour. Our auditive system is built to recognise the absolute, not relative position of the formants (whereas the spectrum of a sound refers to the positions of partials relative to the fundamental) as this is the core of our ability to recognise speech – and be able to communicate even with the opposite sex. The characteristic colour of a stringed instrument is defined by the shape of these formants which in turn depends on the strings' Young's modulus.

6 Conclusion

In this paper, I showed the role of the Young's modulus should not be restricted to its influence over inharmonicity: just as importantly, it is a fundamental factor in sound colour which should be carefully taken into account when faithfully restoring an instrument. This paper describes an easy technique to measure the Young's modulus *in situ* and thus give an interesting piece of information about the acoustical properties of the material of an existing string, useful when choosing replacement strings.